ANOTHER HOMOGENEOUS PLANE CONTINUUM

BY

R. H. BING AND F. B. JONES

1. **History of problem.** A set X is homogeneous if for each pair of points x, y of X there is a homeomorphism of X onto itself that takes x into y.

In 1920 Knaster and Kuratowski [21] raised the following question: If a nondegenerate bounded plane continuum is homogeneous, is it necessarily a simple closed curve?

In 1922, Knaster [20] described a hereditarily indecomposable plane continuum. It is reported that he suspected that this continuum had other interesting properties and suggested to his students the problem of discovering if this Knaster continuum (as it came to be called) had the property possessed by an arc of being topologically equivalent to each of its nondegenerate subcontinua. This Knaster continuum is homogeneous and furnishes a counterexample to an affirmative answer of the above question, but this was not discovered until 1951.

A partial affirmative solution was given to the question in 1924 when Mazurkiewicz [22] showed that the bounded nondegenerate homogeneous plane continuum is a simple closed curve if it is locally connected. This result was improved in 1951 when Cohen [12] showed that the continuum is a simple closed curve if it either is arcwise connected or contains a simple closed curve.

A false affirmative solution was announced [25] in 1937. (Of course, at the time, it was not known that the solution was false—this only developed eleven years later when a counter-example was given.) This false solution was extended [11] in 1944 when an attempt was made to classify all homogeneous bounded closed plane sets.

That a pseudo-arc is homogeneous was shown first by Bing [3] in 1948 and shortly thereafter by essentially the same methods by Moise [24]. Both of these proofs made use of the description of the pseudo-arc given by Moise [23] to show that a pseudo-arc is topologically equivalent to each of its nondegenerate subcontinua.

In 1951 Bing [4] discovered that the pseudo-arc described by Moise in 1948 is actually topologically equivalent to the continuum Knaster described twenty-six years earlier by different methods. In fact, it was shown that any two nondegenerate snake-like hereditarily indecomposable continua are topologically equivalent. Also, it was shown that in the category sense, most

Presented to the Society, September 2, 1954 by R. H. Bing and by F. B. Jones under the title *On a certain type of homogeneous plane continuum;* received by the editors April 15, 1957.

bounded continua in E^n $(n>1)$ are pseudo-arcs. Hence, most bounded plane continua are homogeneous.

In 1953 two papers [18; 19] appeared questioning the homogeneity of the pseudo-arc. The second of the papers made an unsuccessful attempt to correct an error in the first. Inasmuch as these papers received not unfavorable reviews (Mathematical Reviews vol. 15 (1954) p. 146; p. 335) despite the errors in them, it seems desirable to mention that the pseudo-arc has not been abandoned as an example of a homogeneous plane continuum.

In 1954, working independently, Bing and Jones each discovered a homogeneous plane continuum that was neither a simple closed curve nor a pseudo-arc. Neither knew of the others work until the titles of the papers appeared adjacent to each other on the 1954 summer program of the American Mathematical Society [6; 15]. Inasmuch as both had discovered the same example, it was decided to make this a joint paper. The first part of this paper showing that the example—a circle of pseudo-arcs—is homogeneous was written by Bing. The latter part showing that such a circle of pseudo-arcs can be imbedded in the plane was prepared by Jones.

Perhaps the future holds the answer as to whether or not there are other homogeneous bounded plane continua. Jones' result [14] that each bounded homogeneous plane continuum which does not separate the plane is indecomposable may guide our search. Bing has described a continuum (Example 2 of [4]) that is suspected of being homogeneous. However, at the moment, the only nondegenerate bounded plane continua known to be homogeneous are the simple closed curve, the pseudo-arc, and the circle of pseudo-arcs described in this paper. Other pertinent references are found in the bibliography of this paper.

2. **The example.** In this section we describe a homogeneous plane continuum which we call a circle of pseudo-arcs. First we define some terms that we shall use.

A *chain* is a finite collection D of open sets $d_1, d_2, \cdots, d_n$ such that d_i intersects d_j if i, j are adjacent integers and otherwise $\rho(d_i, d_j)$ is positive. We say that i, j are *adjacent* if $|i-j| \leqq 1$—hence i is adjacent to itself. We use ρ to denote the *distance* function. In earlier papers the weaker condition that nonadjacent links did not intersect was used instead of the stronger condition that they were a positive distance apart but in some respects this is less convenient for our present purposes than the stronger condition.

The elements d_1, d_n of the above chain D are called *end links* of D and the other links are called *interior links*. If each link is of diameter less than ϵ, D is called an *ϵ-chain*. The *subchain* of D consisting of $d_r, d_{r+1}, \cdots, d_s$ is denoted by $D(r, s)$.

Suppose $D=d_1, d_2, \cdots, d_n$ is a chain which *covers* a point set X. Then each point of X lies in some link of D. We say that D *properly covers* X if each link of D contains a point of X. Also, D *irreducibly covers* X of each link of D contains a point of X that is not covered by any other link of D. If a chain irreducibly covers a set, then it properly covers the set, but not conversely.

A compact continuum is called *snake-like* [5] if for each positive number ϵ, it can be covered by an ϵ-chain. The *pseudo-arc* described in [23; 3; 4; 7] is an example of a snake-like continuum.

A continuum G is called an *arc of continua* $\{g_x | 0 \leqq x \leqq 1\}$ if there is a map f of G onto $[0, 1]$ such that $f^{-1}(x)$ is the continuum g_x. Then $\{g_x\}$ is an upper semicontinuous collection of continua filling G such that the corresponding decomposition space is an arc. Then g_0 and g_1 are called the *end elements* of $\{g_x\}$. We use $G(a, b)$ to denote the sum of all continua in the collection $\{g_x | a \leqq x \leqq b\}$.

Such a continuum G is a *snake-like arc of continua* $\{g_x | 0 \leqq x \leqq 1\}$ if it is snake-like. If each element of $\{g_x\}$ is a pseudo-arc, G is called a *snake-like arc of pseudo-arcs*. If in addition, the collection $\{g_x\}$ is continuous (f is open), then G is called a *continuous snake-like arc of pseudo-arcs*. In Theorem 10 we show that any two continuous snake-like arcs of pseudo-arcs of homeomorphic—indeed there is a very strong type of homeomorphism between them.

A *circular chain* differs from a regular chain in that the first and last links intersect. A compact nonsnake-like continuum which for each positive number ϵ can be covered by an ϵ-circular chain is called *circle-like*.

The bounded plane continuum which is shown in this paper to be homogeneous is a continuous circle-like circle of pseudo-arcs—which we call for brevity, a *circle of pseudo-arcs*. It is a circle-like continuum M such that there is a continuous decomposition of M into pseudo-arcs such that the decomposition space is a simple closed curve. Hence, there is an open map f of M onto a circle such that the inverse of each point of the circle is a pseudo-arc.

Perhaps certain of the continua used by Anderson in [1; 2] were circles of pseudo-arcs but this was not pointed out there. It follows from Theorem 10 that any two circles of pseudo-arcs are topologically equivalent and that each is homogeneous.

In establishing homeomorphisms between snake-like continua it is frequently convenient to consider sequences of chains covering them and relations between these chains. The chain D_2 refines the chain D_1 if each link of D_2 lies in a link of D_1. A *chain map* H of a chain E into a chain D is a single valued function that assigns a link of D to each link of E such that the images of adjacent links are adjacent—that is $H(e_i)$ and $H(e_{i+1})$ are adjacent links of D. If each link of D is the image of a link of E, we say that H maps E *onto* D. We note that the chain map H does not operate on the points in the links of E but only on the links. Chain maps play much the same role in this paper as *following a pattern* did in [3].

3. Snake-like arcs of continua. In this section we prove some theorems about snake-like continua. A modification of the argument in the following proof shows that each tree-like continuum as defined in [5] is unicoherent but we shall not be concerned with tree-like continua in this paper.

THEOREM 1. *Each snake-like continuum is unicoherent.*

Proof. Suppose a snake-like continuum M is the sum of two continua M_1, M_2. We show that M is unicoherent by showing that each pair of points p, q of $M_1 \cdot M_2$ belongs to a component in $M_1 \cdot M_2$.

Suppose $d_1, d_2, \cdots, d_n$ is a chain covering M, $p \in d_i$, $q \in d_j$, and $i<j$. Since each of M_1, M_2 is connected and contains $p+q$, each intersects each link of $d_i, d_{i+1}, \cdots, d_j$. Hence there is a sequence of points $p=p_i, p_{i+1}, \cdots, p_j = q$ where $p_r \in d_r \cdot M_1$. If $d_1, d_2, \cdots, d_n$ is an ϵ-chain, then $\rho(p_r, M_2)<\epsilon$ and $\rho(p_r, p_{r+1})<2\epsilon$.

Since M is snake-like, for each positive number ϵ it can be covered by an ϵ-chain. The results of the preceding paragraph show that there is a sequence $R_1, R_2, \cdots$ such that R_k is a finite number of points $p=p_1^k$, $p_2^k, \cdots, p_t^k=q$ such that $p_r^k \in M_1$, $\rho(p_r^k, M_2)<1/k$, and $\rho(p_r^k, p_{r+1}^k)<1/k$.

Some subsequence of $R_1, R_2, \cdots$ converges to a set L. But L is connected, contains $p+q$, and belongs to both M_1 and M_2. Since p and q are arbitrary points of $M_1 \cdot M_2$, we have that $M_1 \cdot M_2$ is connected and M is unicoherent.

THEOREM 2. *Suppose G is a snake-like arc of continua $\{g_x \mid 0 \leqq x \leqq 1\}$ and $0<a<b<c<d<1$. There is a positive number ϵ such that if D is any ϵ-chain whatever covering G, any link of D intersecting $G(b, c)$ is between any link of D intersecting $G(0, a)$ and any link intersecting $G(d, 1)$.*

Proof. The required number ϵ is any positive number less than each of $\rho(G(0, a), G(b, c))$, $\rho(G(b, c), G(d, 1))$, and $\rho(G(0, b), G(c, 1))/2$.

Let $D=d_1, d_2, \cdots, d_n$ be an ϵ-chain covering G and d_i, d_j, d_k be elements of D intersecting $G(0, a)$, $G(b, c)$, and $G(d, 1)$ respectively where $i<k$. We show that $i<j<k$.

Since each of $G(0, b)$, $G(c, 1)$ is connected, there are subchains D', D'' of D containing d_i and d_k respectively and covering $G(0, b)$, $G(c, 1)$ respectively such that each link of D' intersects $G(0, b)$ and each link of D'' intersects $G(c, 1)$. Since $\rho(G(0, b), G(c, 1))>2\epsilon$, some link d_t of D lies between the links of D' and the links of D''. Hence $i<t<k$.

If $j<i<t$, $G(b, c)$ intersects d_i because $G(b, c)$ is connected and intersects each of d_j and d_t. This is impossible because $G(0, a)$ intersects d_i and $\rho(G(0, a), G(b, c))>\epsilon$. Similarly, it is impossible that $t<k<j$. Hence $i<j<k$.

THEOREM 3. *Suppose G is a snake-like arc of continua $\{g_x \mid 0 \leqq x \leqq 1\}$. Then for each positive number ϵ there is an ϵ-chain D covering G such that the first link of D intersects g_0 and the last link intersects g_1.*

Proof. Suppose δ is a positive number less than each of $\epsilon/2$ and $\rho(g_0, g_1)/2$.

Since G is snake-like, there is a δ-chain $E=e_1, e_2, \cdots, e_n$ covering G. Suppose e_i, e_j are the first and last members of this chain intersecting g_0 while e_r, e_s are the first and last members of the chain intersecting g_1. Then $E(i, j)$ and $E(r, s)$ are subchains of E properly covering g_0 and g_1 respectively and no link of $E(i, j)$ intersects any link of $E(r, s)$. For convenience we suppose that $i<j<r<s$.

Let a, b be numbers such that $0<a<b<1$, $E(i, j)$ covers $G(0, a)$, and $E(r, s)$ covers $G(b, 1)$. An application of Theorem 2 gives a positive number γ such that if F is a γ-chain covering G, then any link of F intersecting $G(a, b)$ lies between any link of F intersecting $G(0, a/2)$ and any intersecting $G((b+1)/2, 1)$. We put the further restriction on γ that if F is a γ-chain irreducibly covering G, then F refines E, any link of F intersecting $G(0, a)$ lies in a link of $E(i, j)$, and any link of F intersecting $G(b, 1)$ lies in a link of $E(r, s)$.

Let $F=f_1, f_2, \cdots, f_m$ be a γ-chain irreducibly covering G and f_t, f_u be links of F intersecting g_0 and g_1 respectively such that no link of F between f_t and f_u intersects g_0+g_1. For convenience we suppose that $t<u$. Let $F(v, w)$ be the subchain of $F(t, u)$ which is maximal with respect to the property that f_v lies in an end link of $E(i, j)$ and f_w lies in an end link of $E(r, s)$. Then $E(i,j)$ covers each link of $F(1, v)$ and $E(r, s)$ covers each link of $F(w, m)$. For convenience we suppose that f_v lies in e_j and f_w lies in e_r.

Use A to denote the sum of the elements of $F(1, v)$ and B to denote the sum of the elements of $F(v, m)$. Then the elements of D are $A\cdot e_i, A\cdot e_{i+1}, \cdots, A\cdot e_j, f_{v+1}, f_{v+2}, \cdots, f_{w-1}, B\cdot e_r, B\cdot e_{r+1}, \cdots, B\cdot e_s$.

THEOREM 4. *Suppose G is a snake-like arc of continua $\{g_x \mid 0\leqq x\leqq 1\}$ and p is a point of $G-(g_0+g_1)$. Then each neighborhood of p which does not intersect g_0+g_1 separates g_0 from g_1 in G.*

Proof. Suppose U is a neighborhood of p which does not intersect g_0+g_1 and ϵ is a positive number such that $\epsilon<\rho(p, G-U)$.

By Theorem 3 there is an ϵ-chain $D=d_1, d_2, \cdots, d_n$ covering G such that d_1 intersects g_0 and d_n intersects g_1. Let d_i be an element of D that contains p. Then $G-U$ is the sum of the mutually exclusive closed sets $G\cdot(d_1+d_2+\cdots+d_i)-U$ and $G\cdot(d_i+d_{i+1}+\cdots+d_n)-U$ while the first of these sets contains g_0 and the second contains g_1.

THEOREM 5. *Suppose G is a snake-like arc of indecomposable continua $\{g_x \mid 0\leqq x\leqq 1\}$ and E is a chain covering G such that E properly covers each g_x. Then if U is an open set such that $U\cdot G\neq 0$, $U\cdot(g_0+g_1)=0$, and $\overline{U}$ lies in a link of E, then $G-U$ may be expressed as the sum of two mutually separated sets A, B such that $g_0\subset A$, $g_1\subset B$, and for each x $(0\leqq x\leqq 1)$, either $A\cdot g_x$ intersects each link of E or $B\cdot g_x$ intersects each link of E.*

Proof. Let $D_1, D_2, \cdots$ be a decreasing sequence of chains covering G such that the first link of each D_i intersects g_0 and the last link intersects g_1.

Theorem 3 shows that there are such chains.

Let d_1 be a link of some D_i such that $d_1 \subset U$ and diameter d_1 is less than 1. Then Theorem 4 implies that $G-d_1$ is the sum of two mutually separated sets A_1, B_1 containing g_0, g_1 respectively. Using the assumption that the theorem is false, we find that there are numbers a_1, b_1 such that $0<a_1<b_1<1$ and for each x $(a_1 \leqq x \leqq b_1)$, neither $A_1 \cdot g_x$ nor $B_1 \cdot g_x$ intersects each link of E.

Let d_2 be a link of some D_i such that $d_2 \subset d_1$, $d_2 \cdot (G(0, a_1)+G(b_1, 1))=0$, and diameter d_2 is less than $1/2$. Then $G-d_2$ is the sum of two mutually separated sets A_2, B_2 containing $G(0, a_1)$ and $G(b_1, 1)$ respectively. If the theorem is false, there are numbers a_2, b_2 such that $a_1<a_2<b_2<b_1$ and for each g_x $(a_2 \leqq x \leqq b_2)$ neither $A_2 \cdot g_x$ nor $B_2 \cdot g_x$ intersects each link of E.

We continue this procedure to get a sequence of open sets $d_1, d_2, d_3, \cdots$, a sequence of numbers $a_1, a_2, a_3, \cdots$ and a sequence of numbers $b_1, b_2, b_3, \cdots$ such that $d_{j+1} \subset d_j$, d_j is of diameter less than $1/j$ and is a link of some D_i, $a_j<a_{j+1}<b_{j+1}<b_j$, and $G-d_j$ is the sum of two mutually separated sets A_j, B_j containing $G(0, a_{j-1})$, $G(b_{j-1}, 1)$ respectively.

Let c be a number such that $0<a_1<a_2< \cdots <c< \cdots <b_2<b_1<1$. We show that the assumption that the theorem is false leads to the contradiction that g_c is decomposable. For some increasing sequence of integers $n(1)$, $n(2), \cdots$, $A_{n(1)} \cdot g_c, A_{n(2)} \cdot g_c, \cdots$ converges to a set A_c and $B_{n(1)} \cdot g_c, B_{n(2)} \cdot g_c, \cdots$ converges to a set B_c. Now A_c is a continuum because $A_i \cdot g_c$ is not the sum of two sets whose distance apart is more than $1/i$. It is a proper subcontinuum of g_c since no $A_i \cdot g_c$ intersects each link of E. Also, B_c is a proper subcontinuum of g_c. But g_c is decomposable because it is the sum of the two proper subcontinua A_c, B_c.

4. **Chain maps.** In this section we give some theorems about chain maps. The first of these might be labeled a fixed point theorem for chain maps.

THEOREM 6. *Suppose* $D=d_1, d_2, \cdots, d_n$ *and* $E=e_1, e_2, \cdots, e_m$ *are chains and* H_1, H_2 *are two chain maps of* D *into* E *such that* H_i $(i=1, 2)$ *takes a link of* D *into* e_{m-1}. *Then for some link* d_i *of* D, *the link* $H_1(d_i)$ *of* E *is adjacent to the link* $H_2(d_i)$.

Proof. Suppose $H_1(d_1)$ precedes $H_2(d_1)$ in $e_1, e_2, \cdots, e_m$. If each link $H_1(d_j)$ of E precedes the corresponding link $H_2(d_j)$, let d_i be a link of D such that $H_1(d_i)=e_{m-1}$. Then $H_1(d_i)$ is adjacent to $H_2(d_i)$.

If for some link d_j of D, $H_1(d_j)$ does not precede $H_2(d_j)$, let d_i be the first such link of D. Then $H_1(d_i)$ is adjacent to $H_2(d_i)$.

We note that if one chain covers a pseudo-arc, then another chain covering it can be inscribed in the first in a prescribed way.

THEOREM 7. *Suppose* $D=d_1, d_2, \cdots, d_n$ *is a chain properly covering a pseudo-arc* P *and* H *is a chain map of a chain* $X=x_1, x_2, \cdots, x_m$ *onto* D. *Then there is a chain* $E=e_1, e_2, \cdots, e_m$ *properly covering* P *such that* $e_i \subset d_j$ *if* $H(x_i)=d_j$.

In fact, if A, B *are closed sets in* $P\cdot d_1$, $P\cdot d_n$ *respectively and* $H(x_r)=d_1$, $H(x_s)=d_n$, *the chain* E *may be selected so that* $A\subset e_r$, $B\subset e_s$.

Proof. The proof would be slightly easier if $A\subset d_1-\bar{d}_2$, $B\subset d_n-\bar{d}_{n-1}$, $r=1$, $s=m$, x_r is the only element of X that H takes into d_1, and x_s is the only element of X that H takes into d_n. We now prove the theorem in this special case.

Let p_1, p_2 be points of different components of P in d_1-d_2, d_n-d_{n-1} respectively and $D_1, D_2, \cdots$ be a sequence of chains from p_1 to p_2 such that each D_i covers P, D_i is of mesh less than $1/i$, D_{i+1} is crooked in D_i, and $D=D_1$. Then it follows from Theorem 6 of [3] that there is a chain $E=e_1$, $e_2, \cdots, e_m$ from p_1 to p_2 such that E covers P, $e_i\subset d_j$ if $H(x_i)=d_j$, and for some integer r, each link of E is the sum of links of D_r.

Now that we have shown the theorem is true in the special case we alter D, X, H and obtain D', X', H' so that the special case applies. Then we adjust the E' obtained to get the required E. We suppose with no loss of generality that $r<s$.

The chain D' is obtained as follows. Let U_0, U_1, U_2 be open subsets of d_1, $d_1\cdot d_2$, d_2 respectively such that $P\cdot(d_1+d_2)\subset U_0+U_1+U_2$, $P\cdot(d_1-d_2)+A\subset U_0$, $P\cdot(d_2-d_1)\subset U_2$, $U_0\cdot\overline{U}_2=0$, and $A\cdot\overline{U}_1=0$. Also, let U_{n-1}, U_n, U_{n+1} be open subsets of d_{n-1}, $d_{n-1}\cdot d_n$, d_n respectively such that $P\cdot(d_{n-1}+d_n)\subset U_{n-1}+U_n+U_{n+1}$, $P\cdot(d_{n-1}-d_n)\subset U_{n-1}$, $P\cdot(d_n-d_{n-1})+B\subset U_{n+1}$, $\overline{U}_{n-1}\cdot\overline{U}_{n+1}=0$, and $\overline{U}_n\cdot B=0$. Then $D'=U_0, U_1, U_2, d_3, \cdots, d_{n-2}, U_{n-1}, U_n, U_{n+1}$.

The chain $X'=y', y_r, y_{r-1}, \cdots, y_2, x_1, x_2, \cdots, x_m, z_{m-1}, \cdots, z_s, z'$ is obtained from X by adding r elements on at the front and $m+1-s$ on at the end. Then H' is the chain map of X' onto D' so that $H'(y')=U_0$, $H'(z')=U_{n+1}$, $H'(x_i)=H'(y_i)=H'(z_i)=U_j$ (or d_j) if $H(x_i)=d_j$.

Since the special case applies to D', X', and H', there is a chain $E'=f'$, $f_r, f_{r-1}, \cdots, f_2, e_1', e_2', \cdots, e_m', g_{m-1}, \cdots, g_s, g'$ such that $A\subset f'$, $B\subset g'$, and a link e' of E' lies in a link d of D' provided d is the image under H' of the link of X' corresponding to e'.

The chain $E=e_1, e_2, \cdots, e_m$ satisfying the conclusions of the theorem is obtained by adding together certain elements of E'. The link e_r is the sum of f', f_r, e_r' while e_s is the sum of g', g_s, e_s'. In general e_i is the sum of e_i', f_i, g_i (if there are such f's and g's).

The following theorem concerning the existence of chains that cover a snake-like arc of pseudo-arcs in a prescribed fashion will be of use in showing that certain such continua are homeomorphic.

THEOREM 8. *Suppose* P *is a snake-like arc of pseudo-arcs* $\{p_x \mid 0\leqq x\leqq 1\}$, $D=d_1, d_2, \cdots, d_n$ *is a chain covering* P *such that each link of* D *intersects each* p_x, *and* H *is a chain map of a chain* $Y=y_1, y_2, \cdots, y_m$ *onto* D. *Then there is a chain* $E=e_1, e_2, \cdots, e_m$ *covering* P *such that each* p_x *intersects each link of* E *and* $e_i\subset d_j$ *if* $H(y_i)=d_j$.

Proof. First we consider the case where $H(y_1)=d_1$, $H(y_m)=d_n$. Let U_A, U_B be open sets with closures in d_1, d_2 respectively such that U_A, U_B intersect each element of $\{p_x\}$.

It follows from Theorem 7 that for each a $(0\leqq a\leqq 1)$ there is a chain $E(a)=e(a)_1, e(a)_2, \cdots, e(a)_m$ covering p_a such that $\overline{U}_A\cdot p_a\subset e(a)_1$, $\overline{U}_B\cdot p_a\subset e(a)_m$, $e(a)_i\subset d_j$ if $H(y_i)=d_j$.

The upper semicontinuity of the collection $\{p_x\}$ implies that a lies in a connected open subset I_a of $(0\leqq x\leqq 1)$ such that if $x\in I_a$, then $E(a)$ covers p_x, $\overline{U}_A\cdot p_x\subset e(a)_1$, and $\overline{U}_B\cdot p_x\subset e(a)_m$. A finite collection $I_{a_1}, I_{a_2}, \cdots, I_{a_t}$ of such open connected subsets cover $(0\leqq x\leqq 1)$. Let $E(a_1), E(a_2), \cdots, E(a_t)$ be the corresponding $E(a)$'s.

We now show how to form E from $E(a_1), E(a_2), \cdots$ $E(a_t)$. It follows from Theorem 4 that P is covered by open sets $U_1, U_2, \cdots, U_t$ such that $U_i\cdot U_j=U_A+U_B$ if $i\neq j$ and if $p_x\cdot(U_i-(U_A+U_B))\neq 0$, then $x\in I_{a_i}$. Then the chain E whose kth link $e_k=e(a_1)_k\cdot U_1+e(a_2)_k\cdot U_2+\cdots+e(a_t)_k\cdot U_t$ is the required chain.

The purpose of the restriction that $H(y_1)=d_1$, $H(y_m)=d_n$ was to make it apparent that each p_x intersected each link of E. To prove the theorem in the more general case, suppose $H(y_r)=d_1$, $H(y_s)=d_n$. Then alter Y as was done in the proof of Theorem 7 by adding elements onto the first of it and elements onto the last of it so that the resulting chain Y' has ends in d_1, d_n respectively and such that the added elements of Y' can be combined with the original links to get a chain resembling Y. Then end chains of the resulting chain E' may be added onto the center of E' as was done in the proof of Theorem 7 so as to get a chain E satisfying the conditions of the theorem.

In stating the following corollary to Theorem 8, the same symbols are used as appear at the place in the proof of Theorem 10 where it is applied.

COROLLARY TO THEOREM 8. *Suppose $Q(i/k, (i+1)/k)$ is a snake-like arc of pseudo-arcs $\{q_x/i/k\leqq x\leqq(i+1)/k\}$, E is a chain covering $Q(i/k, (i+1)/k)$ such that each q_x intersects each link of E, and H_1 is a chain map of a chain $D(r, s)$ onto E. Then there is a positive number ϵ such that if $F(t, u)$ is an ϵ-chain irreducibly covering $Q(i/k, (i+1)/k)$, then there is a chain map $R_{(2i+1)/2k}$ of $F(t, u)$ onto $D(r, s)$ such that for each link f_i of $F(t, u)$, $f_i\subset H_1R_{(2i+1)/2k}(f_i)$, and if $f_v, f_{v+1}, \cdots, f_w$ is a subchain of $F(t, u)$ irreducibly covering some q_x, then each link of $D(r, s)$ is the image of three consecutive links of $f_v, f_{v+1}, \cdots, f_w$.*

In obtaining this result from Theorem 8, we let the P, $\{p_x|0\leqq x\leqq 1\}$, and D of the theorem be the $Q(i/k, (i+1)/k)$, $\{q_x|i/k\leqq x\leqq(i+1)/k\}$, and E of the corollary, Y be a chain with four times as many links as $D(r, s)$, and H be a chain map that takes links of Y numbered $4i+1$, $4i+2$, $4i+3$, $4i+4$ into the image under H_1 of the ith link of $D(r, s)$. Then ϵ is a number so small that any ϵ-chain properly covering $Q(i/k, (i+1)/k)$ is a refinement of the chain E promised by Theorem 8.

5. **A homeomorphism between pseudo-arcs.** The following theorem shows that if the same ϵ-chain properly covers two pseudo-arcs, there is a homeomorphism of one onto the other that moves no point by more than ϵ. It also shows how a chain map can be approximated with a homeomorphism.

THEOREM 9. *Suppose D, E are chains properly covering pseudo-arcs P, Q and H is a chain map of D onto E such that each link of E is the image of a link of D which contains a point of P not on the closure of any other link of D. Then there is a homeomorphism h of P onto Q such that for each link d_i of D, $h(P\cdot d_i) \subset H(d_i)$.*

The rather awkward condition that each link of E is the image of a link of D that covers a point of P not covered by the closure of any other link of D is necessary because it may be that $D=d_1, d_2, \cdots, d_n$ and $E=e_1, e_2, \cdots, e_n$ have the same number of links, $H(d_i)=e_i$, $P\subset \bar{d}_2+\bar{d}_3+\cdots+\bar{d}_{n-1}$, and $Q \not\subset \bar{e}_2+\bar{e}_3+\cdots+\bar{e}_{n-1}$. The less complicated but more stringent condition that each link of E is the image of an interior link of D might be substituted.

We use two simplifications in this proof so as to make the remaining part of the proof essentially like the proof of Theorem 12 of [3].

FIRST SIMPLIFICATION. We now show that there is no loss of generality in supposing that D and $E=e_1, e_2, \cdots, e_n$ have the same number of links, H takes the ith link of D into the ith link of E, and each end link of D contains a point not on the closure of any other link of D.

Let $D''=d_1'', d_2'', \cdots, d_n''$ be the chain such that d_i'' is the sum of the links of D that go into the ith link of E under H. Then D'' is a chain covering P such that each end link of D'' contains a point not contained on the closure of any other link of D''. Also, if h is a homeomorphism of P onto Q such that $h(P\cdot d_i'')\subset e_i$, then h is the required homeomorphism. Hence we suppose with no loss of generality that $D''=d_1'', d_2'', \cdots, d_n''=d_1, d_2, \cdots, d_n=D$, $H(d_i)=e_i$, $P\cdot(d_1-\bar{d}_2)$ contains a point p_1, and $P\cdot(d_n-\bar{d}_{n-1})$ contains a point p_2 such that p_1, p_2 belong to different composants of P.

SECOND SIMPLIFICATION. We replace chains $D=d_1, d_2, \cdots, d_n$ and $E=e_1, e_2, \cdots, e_n$ left after the first simplification with chains $D'=d_1', d_2', \cdots, d_{4n-3}'$ and $E'=e_1', e_2', \cdots, e_{4n-3}'$ covering P and Q respectively such that $p_1\in(d_1'-\bar{d}_2')$, $p_2\in(d_{4n-3}'-\bar{d}_{4n-4}')$, and if h is any homeomorphism of P onto Q such that $h(P\cdot d_i')\subset \bar{e}_{i-1}'+\bar{e}_i'+\bar{e}_{i+1}'$, then h is the required homeomorphism.

First we suppose that q_1, q_2 are points of different composants of Q in e_1-e_2 and e_n-e_{n-1} respectively. If there are not already such points, points may be deleted from e_2 and e_{n-1} to make the condition satisfied.

Suppose that ϵ is a positive number so small that each subset of Q of diameter less than 5ϵ lies in one link of E. Let F be an ϵ-chain properly covering Q. Then e_1' is the sum of all links of F whose closures intersect e_1-e_2, e_5' is the sum of all links of F whose closures intersect $e_2-(e_1+e_3)$, e_9' is the sum of all links of F whose closures intersect $e_3-(e_2+e_4)$, $\cdots$, and e'_{4n-3} is the sum of all links of F whose closures intersect e_n-e_{n-1}. The links of F with closures in $e_1\cdot e_2$ are combined to form e'_2, e'_3, e'_4; the links of F with closures in $e_2\cdot e_3$ are combined to form e'_6, e'_7, e'_8, $\cdots$, and the links of G with closures in $e_{n-1}\cdot e_n$ are combined to form e'_{4n-6}, e'_{4n-5}, e'_{4n-4}. (See Figure 1.)

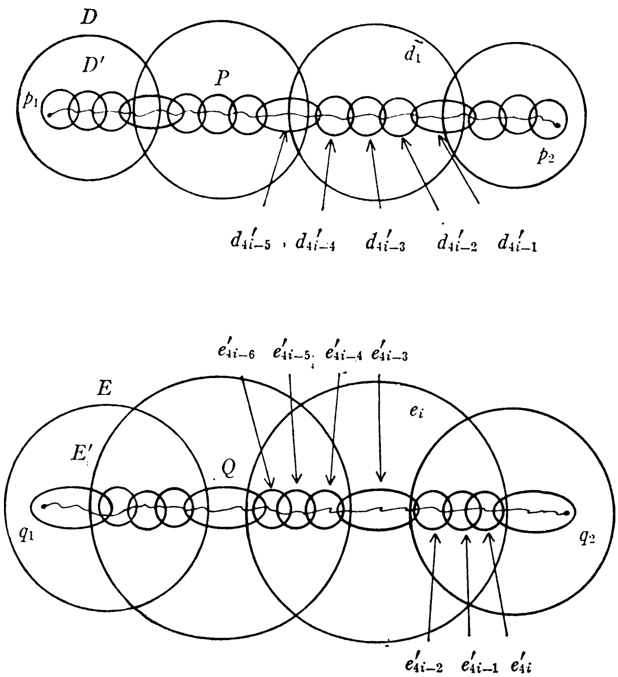


FIG. 1

Let δ be a positive number so small that the distance between any pair of nonadjacent links of $D=d_1, d_2, \cdots, d_n$ is more than 5δ and $\rho(p_1+p_2, d_2+d_{n-1})>3\delta$. Let G be a δ-chain properly covering P and refining D. Then d'_3 is the sum of the links of G that intersect $d_1\cdot d_2$, d'_7 is the sum of the links of G that intersect $d_2\cdot d_3$, $\cdots$, and d'_{4n-5} is the sum of the links of G that intersect $d_{n-1}\cdot d_n$. Also, d'_1 is the sum of the links of G whose closures contain p_1; d'_2 is the sum of the other links of G in e_1-e_2; d'_4, d'_5, d'_6 are formed by combining links of G in $e_2-(e_1+e_3)$, $\cdots$, d'_{4n-3} is the sum of the links of G whose closures contain p_2; d'_{4n-4} is the sum of the other links of G in e_n-e_{n-1}. See Figure 1.

We now show that if h is a homeomorphism of P onto Q such that $h(P\cdot d'_i)\subset\bar{e}'_{i-1}+\bar{e}'_i+\bar{e}'_{i+1}$, then h is the required homeomorphism because $h(P\cdot d_i)\subset e_i$. The reason is that $h(P\cdot d_i)\subset h(P\cdot(d'_{4i-5}+d'_{4i-4}+d'_{4i-3}+d'_{4i-2}+d'_{4i-1}))\subset \bar{e}'_{4i-6}+\bar{e}'_{4i-5}+\bar{e}'_{4i-4}+\bar{e}'_{4i-3}+\bar{e}'_{4i-2}+\bar{e}'_{4i-1}+\bar{e}'_{4i}\subset\bar{e}_i$.

Completion of proof. It follows from Theorem 13 of [3] and the definition of a pseudo-arc that there is a sequence of chains $F_1, F_2, \cdots$ from p_1 to p_2 such that F_i covers P, F_{i+1} is crooked in F_i, and F_i is of mesh less than $1/i$. Also, there is a sequence of chains $G_1, G_2, \cdots$ from q_1 to q_2 such that G_i covers Q, G_{i+1} is crooked in G_i, and G_i is of mesh less than $1/i$.

In order to define the homeomorphism h we shall obtain a sequence of chains $D_1, D_2, \cdots$ from p_1 to p_2 and a sequence of chains $E_1, E_2, \cdots$ from q_1 to q_2 such that (1) D_i covers P and E_i covers Q; (2) D_i, E_i $(i>2)$ are of mesh less than $1/i$; and (3) $D_i=d_1^i, d_2^i, \cdots, d_{n_i}^i$ and $E_i=e_1^i, e_2^i, \cdots, e_{n_i}^i$ have the same number of links and there is a chain map H_i of E_{i+1} onto E_i and D_{i+1} onto D_i such that $d_j^{i+1}\subset H_i(d_j^{i+1})$, $e_j^{i+1}\subset H_i(e_j^{i+1})$, and if $H_i(d_j^{i+1})=d_k^i$, then $H_i(e_j^{i+1})=e_k^i$.

We set $D_1=D'$ and $E_1=E'$ where D' and E' are the chains obtained in the second simplification.

Let E_2 be any G_i of mesh less than 1/3 that refines E_i and H_2 be any chain map of E_2 onto E_1 such that $e_j^2\subset H_1(e_j^2)$ for each j. Then by Theorem 6 of [3] we find that there is a chain D_2 from p_1 to p_2 covering P such that for some integer r, each link of D_2 is the sum of links of F_r, D_2 has the same number of links as E_2, and $d_j^2\subset d_k^1=H_1(d_j^2)$ if $H_1(e_j^2)=d_k^1$.

Let D_3 be any F_i of mesh less than 1/4 that refines D_2 and H_2 be any chain map of D_3 onto D_2 such that $d_j^3\subset H_2(d_j^3)$ for each j. Again it follows from Theorem 6 of [3] that there is a chain E_3 from q_1 to q_2 covering Q such that for some integer r, each link of E_3 is the sum of links of G_r, E_3 has the same number of links as D_3, and $e_j^3\subset e_k^2=H_2(e_j^3)$ if $H_2(d_j^3)=d_k^2$.

The sequences $D_1, D_2, \cdots$ and $E_1, E_2, \cdots$ are obtained by a repetition of this procedure.

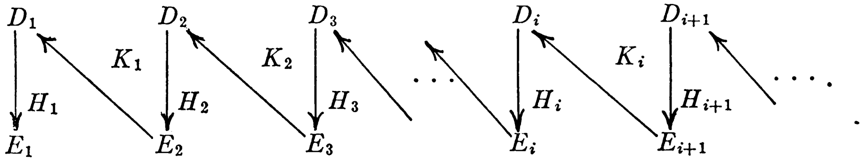
Let $d(p, i)$ denote the sum of the elements of D_i containing the point p of P and $e(p, i)$ denote the sum of the corresponding elements of E_i. Then $d(p, i+1)\subset d(p, i)$. Also, $e(p, i+1)\subset e(p, i)$ since if e_j^{i+1} lies in $e(p, i+1)$, d_j^{i+1} contains p, $H_i(d_j^{i+1})=d_k^i$ contains p and lies in $d(p, i)$, and e_k^i lies in $e(p, i)$ and contains e_j^{i+1}. The homeomorphism h is defined so that $h(p)$ is the intersection of the closures of the decreasing sequence of open sets $e(p, 1), e(p, 2), \cdots$. That h is a homeomorphism of P onto Q follows from an argument similar to that contained at the end of the proof of Theorem 10 of this paper or in Theorem 11 of [3].

Finally we show that $h(P\cdot d_i^1)\subset \bar{e}_{i-1}^1+\bar{e}_i^1+\bar{e}_{i+1}^1$. If p is a point of d_i^1, $d(p, 1)\subset d_{i-1}^1+d_i^1+d_{i+1}^1$ and $h(p)\in \bar{e}(p, 1)\subset \bar{e}_{i-1}^1+\bar{e}_i^1+\bar{e}_{i+1}^1$. Hence $h(P\cdot d_i^1)\subset \bar{e}_{i-1}^1+\bar{e}_i^1+\bar{e}_{i+1}^1$.

6. **Homeomorphisms between arcs of pseudo-arcs.** The theorem of this section shows that there is a strong type of topological equivalence between any two continuous snake-like arcs of pseudo-arcs. It is a key theorem to the showing that each circle of pseudo-arcs is homogeneous.

THEOREM 10. *Suppose P, Q are continuous snake-like arcs of pseudo-arcs $\{p_x\}$, $\{q_x\}$ $(0\leqq x\leqq 1)$. Then each homeomorphism h that takes the sum of the* *ends of P onto the sum of the ends of Q may be extended to a homeomorphism taking P onto Q. In fact, if $h(p_0)=q_0$, the extended homeomorphism h may be chosen so that $h(p_x)=q_x$.*

Proof. Here is an outline of our plan. We shall get a sequence of chains $D_1, D_2, \cdots$ covering P and a sequence of chains $E_1, E_2, \cdots$ covering Q and use these sequences to define our homeomorphism h. Also, we obtain sequences of chain maps $H_1, H_2, \cdots$ and $K_1, K_2, \cdots$ such that H_i takes D_i onto E_i and K_i takes E_{i+1} onto D_i. This is illustrated as follows:



The chain map H_1 will be an approximation to the homeomorphism h we are seeking in that for each point p of P, there is an integer i such that p belongs to the ith link d_i^1 of D_1 and $h(p)$ belongs to the link $H_1(d_i^1)$ of E_1. Also, K_1 will be an approximation to h^{-1} in that for each point q of Q there is an integer i such that q is an element of the ith link e_i^2 of E_2 and $h^{-1}(q)$ is in the link $K_1(e_i^2)$ of D_1. Furthermore, K_1 agrees with H_1^{-1} in that each link of E_2 lies in its image under H_1K_1.

The D's, E's, H's, K's will be defined in the following order: E_1, D_1, H_1, E_2, K_1, D_2, H_2, E_3, K_3, $\cdots$. The meshes of D_i and E_i are less than $1/2^i$. In general, e_j^{i+1} is contained in $H_iK_i(e_j^{i+1})$ where e_j^{i+1} is the jth link of E_{i+1} and d_j^{i+1} is contained in $K_iH_{i+1}(d_j^{i+1})$. (See Figure 2.)

The D's, E's, H's and K's will be chosen so that for each point p of p_x, there is a decreasing sequence of links $d_{n_1}^1, d_{n_2}^2, \cdots$ containing p ($d_{n_i}^i \in D_i$) such that $K_{i-1}H_i(d_{n_i}^i)=d_{n_{i-1}}^{i-1}$ and $H_1(d_{n_1}^1), H_2(d_{n_2}^2), \cdots$ is a sequence of links ($H_i(d_{n_i}^i)\in E_i$) whose closures contain a point q of q_x. The homeomorphism h is chosen so that $h(p)=q$. In order to define the D's, E's, H's, and K's properly, we extend the homeomorphism h to certain elements of $\{p_x\}$ as we go.

We shall use $P(a, b)$ to denote the sum of the elements of $\{p_x \mid a \leqq x \leqq b\}$ and $Q(a, b)$ to denote the sum of the elements of $\{q_x \mid a \leqq x \leqq b\}$.

Now for the details of the proof.

Step 1. In this step we define E_1 and extend the homeomorphism h to some more elements of $\{p_x\}$. The step contains three parts.

(a) Consider a chain E_1 irreducibly covering Q and of mesh less than $1/2$.

(b) It follows from the continuity of the collection $\{p_x\}$ and the bicompactness of an arc that there is a positive integer n such that if $0 \leqq b-a \leqq 1/n$, then the subchain of E_1 that irreducibly covers $Q(a, b)$ properly covers each element of $\{q_x/a \leqq x \leqq b\}$.

(c) Extend the homeomorphism h already defined on p_0+p_1 to $p_0+p_{1/n}$

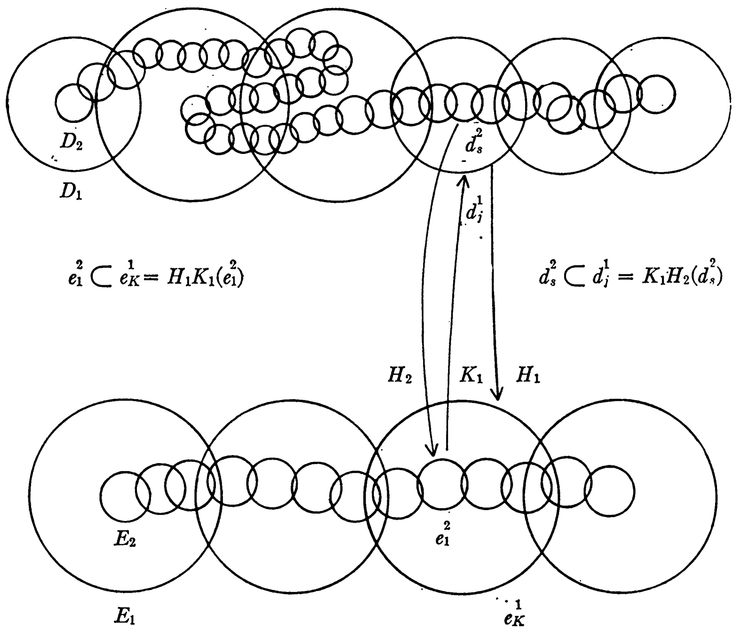


FIG. 2

$+p_{2/n}+ \cdots +p_1$. We do not need to exercise the care in making this extension that we will need to do in later steps. We merely impose the condition that h takes $p_{i/n}$ homeomorphically onto $q_{i/n}$.

Step 2. In this step we define D_1, H_1, and extend the map h still further by defining it on more elements of $\{p_x\}$. We wish the following conditions to be satisfied.

I. If link d_j of D_1 intersects $p_{i/n}$, then $H_1(d_j)$ contains $h(d_j \cdot p_{i/n})$.

II. For each x, H_1 takes any subchain of D_1 that properly covers p_x onto a subchain of E_1 that properly covers q_x.

Parts (a)–(e) of Step 2 are used in getting D_1 and H_1. Parts (f)–(h) show that H_1 and D_1 satisfy the above Conditions I and II. In Part (i) we extend the homeomorphism to other elements of $\{p_x\}$.

(a). In Step 2a we get a preliminary approximation F to the chain D_1. We let F be an ϵ-chain covering P where ϵ is small enough to make the following statements true. (1) $\epsilon<1/2$. (2) No link of F intersects both p_a and p_b if $1/2n \leqq b-a$. (3) The image under h of any subset of $p_0+p_{1/n}+ \cdots +p_1$ of diameter less than 5ϵ lies in an element of E_1. (4) Any subchain of F covering an element of $\{p_x\}$ has at least six times as many links as E_1.

(b) In this part of Step 2 we get approximations R_0, $R_{1/2n}$, $R_{1/n}$, $R_{3/2n}, \cdots$, R_1 to H_1.

First we describe $R_{i/n}$ $(i=0, 1, \cdots, n)$. Consider the subchain $f_t, f_{t+1}, \cdots, f_u$ of F irreducibly covering $p_{i/n}$ and the subchain e_r, $e_{r+1}, \cdots$, e_s of E_1 irreducibly covering $q_{i/n}$. Then $R_{i/n}$ is a chain map of $f_t, f_{t+1}, \cdots, f_u$ onto $e_r, e_{r+1}, \cdots, e_s$ such that $h(f_j \cdot p_{i/n}) \subset R_{i/n}(f_j)$ and for each element e_k of $e_r, e_{r+1}, \cdots, e_s$ there are three consecutive elements of $f_t, f_{t+1}, \cdots, f_u$ that go into e_k under $R_{i/n}$. A precaution to take to insure that each e_k is the image of three consecutive elements of $f_t, f_{t+1}, \cdots, f_u$ is to decide that $R_{i/n}(f_{j-2}) = R_{i/n}(f_{j-1}) = R_{i/n}(f_j) = R_{i/n}(f_{j+1}) = R_{i/n}(f_{j+2}) = e_k$ if e_k is the only element of E_1 such that $h(f_j \cdot p_{i/n}) \subset e_k$. Condition 3 of Step 2a enables us to do this. Unless f_j is the first or last element of F, such an $R_{i/n}$ sends more than three consecutive elements of F into e_k.

Now we describe $R_{(2i+1)/2n}$ $(i=0, 1, \cdots, n-1)$. Let $f_t, f_{t+1}, \cdots, f_u$ and $e_r, e_{r+1}, \cdots, e_s$ be the subchains of F and E_1 that irreducibly cover $P(i/n, (i+1)/n)$ and $Q(i/n, (i+1)/n)$ respectively. It may be that the subscripts t and u mentioned here may differ considerably from the t and u mentioned in the last paragraph but the r and s used here do not differ by more than 1 from those used there. It follows from Step 1b that $e_r, e_{r+1}, \cdots, e_s$ properly covers each element of $\{q_x / i/n \leqq x \leqq (i+1)/n\}$. Then $R_{(2i+1)/2n}$ is any chain map whatever of $f_t, f_{t+1}, \cdots, f_u$ onto $e_r, e_{r+1}, \cdots, e_s$ so long as it is true that for each element e_k of $e_r, e_{r+1}, \cdots, e_s$ and each subchain $f_v, f_{v+1}, \cdots, f_w$ of $f_t, f_{t+1}, \cdots, f_u$ irreducibly covering an element of $\{p_x | i/n \leqq x \leqq (i+1)/n\}$ there are three consecutive elements of $f_v, f_{v+1}, \cdots, f_w$ that go into e_k under $R_{(2i+1)/2n}$. We must be more careful in describing the analogue of $R_{(2i+1)/2n}$ in Step 3 but we can accomplish the result here by letting $R_{(2i+1)/2n}$ send the first three elements of $f_t, f_{t+1}, \cdots, f_u$ into e_r; the next three into $e_{r+1}, \cdots$, the next three into e_s, the next three into $e_{s-1}, \cdots$. Condition 4 of Step 2a assures us that $f_v, f_{v+1}, \cdots, f_w$ has enough elements that such a procedure will cause each element of $e_r, e_{r+1}, \cdots, e_s$ to be the image of some three consecutive members of $f_v, f_{v+1}, \cdots, f_w$ under $R_{(2i+1)/2n}$.

If it were true that all the R's agreed, we could use F and an extension of the R's for D_1 and H_1. However, there is no reason why they must agree so we get a chain D_1 that refines F and a chain map H_1. In obtaining H_1 we shall be influenced by $R_{i/n}$ near $p_{i/n}$ and by $R_{(2i+1)/2n}$ on the part of P between $p_{i/n}$ and $p_{(i+1)/n}$.

(c) We mentioned that we are influenced by $R_{i/n}$ near $p_{i/n}$. We now describe the part P_i of P near $p_{i/n}$ where we were influenced.

Let ϵ be a positive number less than $1/2n$ and so small that if $f_t, f_{t+1}, \cdots, f_u$ is the subchain of F irreducibly covering $p_{i/n}$ and $e_r, e_{r+1}, \cdots, e_s$ is the image of $f_t, f_{t+1}, \cdots, f_u$ under $R_{i/n}$, then $f_t, f_{t+1}, \cdots, f_u$ properly covers each element of $\{p_x | \, |x-i/n| \leqq \epsilon\}$ and $e_r, e_{r+1}, \cdots, e_s$ properly covers each element of $\{q_x | \, |x-i/n| \leqq \epsilon\}$. We shall describe P_i so that $P_i \subset P(i/n-\epsilon, i/n+\epsilon)$.

Since each of $R_{i/n}$, $R_{(2i+1)/2n}$, and $R_{(2i-1)/2n}$ takes $f_t, f_{t+1}, \cdots, f_u$ (and possibly more) onto a subchain (possibly a different one) of E_1 that properly covers $q_{i/n}$, it follows from Theorem 6 that there are an f_y and an f_z of f_t,

$f_{t+1}, \cdots, f_u$ such that $R_{(2i+1)/n}(f_y)$, $R_{i/n}(f_y)$ are adjacent and $R_{(2i-1)/2n}(f_z)$, $R_{i/n}(f_z)$ are adjacent.

By Theorem 5 there is an open set U_i in $P(i/n, i/n+\epsilon)\cdot f_y$ such that $\overline{U}_i \subset f_y$ and $P-U_i$ is the sum of two mutually exclusive closed sets A_i, B_i such that $P(0, i/n)\subset A_i$, $P(i/n+\epsilon, 1)\subset B_i$, and for each element p_x of $\{p_x \mid i/n \leqq x \leqq i/n+\epsilon\}$, there is one of the sets $p_x\cdot A_i$, $p_x\cdot B_i$ that intersects each element of $f_t, f_{t+1}, \cdots, f_u$.

Also, there is an open set V_i in $P(i/n-\epsilon, i/n)$ such that $\overline{V}_i\cdot\overline{U}_i=0$, $\overline{V}_i\subset f_z$, and $P-V_i$ is the sum of two mutually exclusive closed sets A_i', B_i' such that $P(0, 1/n-\epsilon)\subset A_i'$, $P(i/n, 1)\subset B_i'$, for each element p_x of $\{p_x \mid i/n-\epsilon \leqq x \leqq i/n\}$ there is one of the sets $p_x\cdot A_i'$, $p_x\cdot B_i'$ that intersects each element of $f_t, f_{t+1}, \cdots, f_u$.

Then $P_i=A_i\cdot B_i'$ if $i=1, 2, \cdots, n-1$. If $i=0$, $P_i=A_i$; if $i=n$, $P_i=B_i'$. We note that $p_{i/n}\subset P_i$.

(d) In this step we define the chain D_1. Let D_1 be a chain irreducibly covering P of mesh so small that (1) if an element d of D_1 intersects P_i, it lies in $P_i+U_i+V_i$ and in a link of the subchain of F that irreducibly covers $p_{i/n}$, (2) if d does not intersect any P_i, then it lies in some $P(i/n, (i+1)/n)$ and in a link of the subchain of F that irreducibly covers $P(i/n, (i+1)/n)$, (3) d does not intersect both U_i and V_i but it lies in f_y or f_z according as it intersects U_i or V_i. We note that if d intersects P_i, it lies in an element of F on which $R_{i/n}$ is defined and otherwise it lies in an element of F on which some $R_{(2i+1)/2n}$ is defined.

(e) We now describe the chain map H_1 of D_1 onto E_1. Suppose d is an element of D_1 that lies in f_k of F. If d intersects the U_i mentioned in Step 2c, then we pick f_k to be f_y; then $H_1(d)$ is $R_{i/n}(f_y)$ or $R_{(2i+1)/2n}(f_y)$ according as d does or does not intersect P_i. If d intersects V_i, then we pick f_k to be f_z; $H_1(d)$ is $R_{i/n}(f_z)$ or $R_{(2i-1)/2n}(f_z)$ according as d does or does not intersects P_i. If d intersects P_i but neither U_i nor V_i, pick f_k to be any link of F on which $R_{i/n}$ is defined; then $H_1(d)=R_{i/n}(f_k)$. If d does not intersect any $V_i+P_i+U_i$, it lies in some $P(i/n, (i+1)/n)$ so pick f_k to be some link of F on which $R_{(2i+1)/2n}(f_k)$ is defined; then $H_1(d)=R_{(2i+1)/2n}(f_k)$.

(f) Here we show that H_1 is a chain map—that is $H_1(d_j)$ is adjacent to $H_1(d_{j+1})$ where d_j, d_{j+1} are adjacent elements of D_1. If both d_j and d_{j+1} intersect P_i, then $H_1(d_j)$ is adjacent to $H_1(d_{j+1})$ because $R_{i/n}$ is a chain map. If one intersects P_i and the other does not, either both intersect U_i or both intersect V_i. If both intersect U_i, both lie in f_y and $R_{i/n}(f_y)$ is adjacent to $R_{(2i+1)/2n}(f_y)$; if both intersect V_i, both lie in f_z and $R_{i/n}(f_z)$ is adjacent to $R_{(2i-1)/2n}(f_z)$. If neither d_j nor d_{j+1} intersect any P_i, then $H_1(d_j)$ is adjacent to $H_1(d_{j+1})$ because each $R_{(2i+1)/2n}$ is a chain map.

(g) In Step 2g we show that the chain map H_1 satisfies Condition I mentioned at the beginning of Step 2.

Suppose d_j intersects $p_{i/n}$, $d_j\subset f_k$, and $H_1(d_j)=R_{i/n}(f_k)$. We find from the definition of $R_{i/n}$ in Step 2b that $h(f_k \cdot p_{i/n}) \subset R_{i/n}(f_k)$. Hence $h(d_j \cdot p_{i/n}) \subset R_{i/n}(f_k) = H_1(d_j)$.

(h) Now we turn to Condition II. Suppose that d_j intersects p_x $(i/n \leqq x \leqq (i+1)/n)$. First we show that $H_1(d_j)$ intersects q_x.

There is an element f_k of F containing d_j such that $H_1(d_j)$ is equal to either $R_{i/n}(f_k)$, $R_{(2i+1)/2n}(f_k)$, or $R_{(i+1)/n}(f_k)$. If $H_1(d_j) = R_{i/n}(f_k)$, $H_1(d_j)$ is a link of the subchain of E_1 irreducibly covering $q_{i/n}$ and hence intersects q_x as a result of conditions in Step 1b; if $H_1(d_j) = R_{(i+1)/n}(f_k)$, $H_1(d_j)$ is a link of the subchain of E_1 irreducibly covering $q_{(i+1)/n}$ and hence intersects q_x; if $H_1(d_j) = R_{(2i+1)/2n}(f_k)$, $H_1(d_j)$ is a link of the subchain of E_1 irreducibly covering $Q(i/n,\ (i+1)/n)$ as noted in Step 2b and hence intersects each element of $\{q_x / i/n \leqq x \leqq (i+1)/n\}$ by Step 1b.

Suppose $d_v, d_{v+1}, \cdots, d_w$ is the subchain of D_1 that irreducibly covers p_x. Now we show that if e_k is an element of the subchain of E_1 irreducibly covering q_x, then there is an element d_j of $d_v, d_{v+1}, \cdots, d_w$ such that $e_k = H_1(d_j)$.

If p_x intersects $P_i + U_i$ it follows from the definition of ϵ in Step 2c that e_k is an element of the subchain $e_r, e_{r+1}, \cdots, e_s$ of E_1 that irreducibly covers $q_{i/n}$.

If $p_x \cdot P_i$ intersects each element of the subchain $f_t, f_{t+1}, \cdots, f_u$ of F that irreducibly covers $p_{i/n}$, consider the three consecutive elements f_{a-1}, f_a, f_{a+1} of $f_t, f_{t+1}, \cdots, f_u$ that go into e_k under $R_{i/n}$ as mentioned in Step 2b. There is a d_j of $d_v, d_{v+1}, \cdots, d_w$ such that $d_j \cdot p_x \cdot P_i$ intersects f_a. Then $H_1(d_j)$ is equal to either $R_{i/n}(f_{a-1})$, $R_{i/n}(f_a)$, or $R_{i/n}(f_{a+1})$ and all three of them are equal to e_k.

In case p_x intersects $P_i + U_i$ but $p_x \cdot P_i$ does not intersect each of $f_t, f_{t+1}, \cdots, f_u$, then $p_x - (U_i + P_i)$ intersects each element of $f_t, f_{t+1}, \cdots, f_u$ by definition of A_i, B_i in Step 2c. Now let f_{a-1}, f_a, f_{a+1} be three consecutive elements of the subchain $f_t, f_{t+1}, \cdots, f_u$ that go into e_k under $R_{(2i+1)/2n}$. There is a d_j of $d_v, d_{v+1}, \cdots, d_w$ such that d_j lies in f_a and intersects $p_x - (U_i + P_i)$. Then $H_1(d_j)$ is either $R_{(2i+1)/2n}(f_{a-1})$, $R_{(2i+1)/2n}(f_a)$, or $R_{(2i+1)/2n}(f_{a+1})$ and all three are equal to e_k.

In case p_x intersects $V_{i+1} + P_{i+1}$, we also find that there is an element d_j of $d_v, d_{v+1}, \cdots, d_w$ such that $e_k = H_1(d_j)$.

In case p_x does not intersect $P_i + U_i + V_{i+1} + P_{i+1}$, we find that e_k is a link of the subchain of E_k that irreducibly covers $Q(i/n,\ (i+1)/n)$. Let $f_t, f_{t+1}, \cdots, f_u$ be the subchain of F that irreducibly covers p_x and f_{a-1}, f_a, f_{a+1} be the three consecutive elements of $f_t, f_{t+1}, \cdots, f_u$ that go into e_k under $R_{(2i+1)/2n}$. Again we find that there is an element d_j of $d_v, d_{v+1}, \cdots, d_w$ in f_a such that $H_1(d_j) = e_k$.

(i) As in Step 1b we find that there is an integer k such that k is a multiple of n and if $0 \leqq b - a \leqq 1/k$, the subchain of D_1 that irreducibly covers $P(a, b)$ properly covers each element of $\{p_x | a \leqq x \leqq b\}$ and the image of this subchain under H_1 properly covers each element of $\{q_x | a \leqq x \leqq b\}$.

We now extend the homeomorphism h to $p_0, p_{1/k}, p_{2/k}, \cdots, p_1$. This is done so that $h(p_{i/k})=q_{i/k}$ and such that for each element d_j of the subchain of D_1 that properly covers $p_{i/k}$, $h(p_{i/k}\cdot d_j)\subset H_1(d_j)$. Theorem 9 assures us that h can be extended in this fashion.

Step 3. In this step we define a chain E_2 covering Q, a chain map K_1 of E_2 onto D_1, and extend the map h to additional elements of $\{p_x\}$. The step is much like Step 2. We are getting an approximation to h^{-1} instead of to h.

We show that there is a 1/4-chain E_2 irreducibly covering Q and a chain map K_1 of E_2 onto D_1 such that

I. If link e_i of E_2 intersects $q_{j/k}$, then $h^{-1}(e_i\cdot q_{j/k})\subset K_1(e_i)$.

II. For each x, K_1 takes any subchain of E_2 properly covering q_x onto a subchain of D_1 properly covering p_x.

III. $e_i\subset H_1K_1(e_i)$ for each link e_i of E_2.

The thing making Step 3 more complicated than Step 2 is Condition III. The proof that there are E_2 and K_1 satisfying Conditions I and II is similar to that showing that there are D_1 and H_1 satisfying Conditions I and II of Step 2.

Here we show how to alter the arguments of Step 2 to get E_2 and K_1 to satisfy Condition III in addition to I and II.

As in Step 2, we get an approximation F to E_2. The F of Step 3 will be an ϵ-chain irreducibly covering Q where ϵ is small enough to make the following statements true. (1) $\epsilon<1/4$. (2) No link of F intesects q_a and q_b if $1/2k<b-a$. (3) The image under h^{-1} of any subset of $q_0+q_{1/k}+\cdots+q_1$ of diameter less than 5ϵ lies in an element of E_1. (4) ϵ is less than the ϵ mentioned in the corollary to Theorem 8 where the $D(r, s)$ there is the subchain of D irreducibly covering $P(i/k,(i+1)/k)$ $(i=0, 1, \cdots, k-1)$ and E is the image of $D(r, s)$ under H_1.

The chain map $R_{i/k}$ we define here is similar to the $R_{i/n}$'s defined in Step 2b in that $R_{i/k}$ takes the subchain $f_t, f_{t+1}, \cdots, f_u$ of F that irreducibly covers $q_{i/k}$ onto the subchain $d_v, d_{v+1}, \cdots, d_w$ of D_1 that irreducibly covers $p_{i/k}$ such that each element of $d_v, d_{v+1}, \cdots, d_w$ is the image of three consecutive elements of $f_t, f_{t+1}, \cdots, f_u$ and for each j, $h^{-1}(q_{i/k}\cdot f_j)\subset R_{i/k}(f_j)$. Operating on both sides of $h^{-1}(q_{i/k}\cdot f_j)\subset p_{i/k}\cdot R_{i/k}(f_j)$ by h we find that $q_{i/k}\cdot f_j\subset h(p_{i/k}\cdot R_{i/k}(f_j))$. Since $h(p_{i/k}\cdot R_{i/k}(f_j))\subset H_1R_{i/k}(f_j)$ by Step 2i, we have that $q_{i/k}\cdot f_j\subset H_1R_{i/k}(f_j)$.

To complete the analogue of Step 2b, we only need to pick a chain map $R_{(2i+1)/2k}$. It takes the subchain $f_t, f_{i+1}, \cdots, f_u$ of F that irreducibly covers $Q(i/k, (i+1)/k)$ onto the subchain $d_r, d_{r+1}, \cdots, d_s$ of D_1 that irreducibly covers $P(i/k, (i+1)/k)$ so that (1) $f_j\subset H_1R_{(2i+1)/2k}(f_j)$ and (2) for each x $(i/k\leqq x\leqq(i+1)/k)$, $R_{(2i+1)/2k}$ takes the subchain $f_v, f_{v+1}, \cdots, f_w$ of F irreducibly covering q_x onto $d_r, d_{r+1}, \cdots, d_s$ so that each element of $d_r, d_{r+1}, \cdots, d_s$ is the image of three consecutive links of $f_v, f_{v+1}, \cdots, f_w$. That there is such a map $R_{(2i+1)/2k}$ follows from the fact that the mesh of the chain

F was taken smaller than the ϵ mentioned in the corollary to Theorem 8.

In the analogue to Step 2c we pick ϵ so small that the Q_i, V_i, U_i we obtain will have the property that if f_j is an element of the subchain of F irreducibly covering $q_{i/k}$, then $(V_i+Q_i+U_i)\cdot f_j\subset H_1R_{i/k}(f_j)$. (We already have that $q_{i/k}\cdot f_j\subset H_1R_{i/k}(f_j)$ from the definition of $R_{i/k}$.)

Now the argument in Step 3 proceeds as the argument in Step 2 through parts c, d, e, f, g, h, and i. The purpose in extending the homeomorphism h in part (i) is to help in setting up Step 4. We complete the discussion of Step 3 by showing that the E_2 we obtained in the analogue of Step 2d satisfies Condition III with respect to the K_1 we obtained in the analogue of Step 2e.

Suppose the link e_j of E_2 intersects Q_i (where Q_i is the analogue of P_i in Step 2c), $e_j\subset f_z$, and $K_1(f_z)=R_{i/k}(f_z)$. Then f_z is a link of the subchain of F irreducibly covering $q_{i/k}$ and $e_j\subset V_i+Q_i+U_i$. Since $(V_i+Q_i+U_i)\cdot f_z\subset H_1R_{i/k}(f_z)$, $e_j\subset H_1K_1(e_j)$.

Suppose e_j does not intersect any Q_i. Then there is an f_z such that $e_j\subset f_z$ and $K_1(e_j)=R_{(2i+1)/2k}(f_z)$. But since $f_z\subset H_1R_{(2i+1)/2k}(f_z)$, $e_j\subset H_1K_1(e_j)$.

Steps 4, 5, $\cdots$. These steps are essentially repetitions of Step 3. We have Conditions A, A′, B, B′ satisfied which are modifications of Conditions II and III of Step 3.

In Step 2n we find a chain D_n irreducibly covering P and a chain map H_n of D_n onto E_n such that the following conditions hold.

A. For each x $(0\leqq x\leqq 1)$, H_n takes any subchain of D_n properly covering p_x onto a subchain of E_n properly covering q_x.

B. $d_i\subset K_{n-1}H_n(d_i)$ for each link d_i of D_n.

In Step $2n+1$ we get a chain E_{n+1} irreducibly covering Q and a chain map K_n of E_{n+1} onto D_n satisfying the following conditions.

A′. For each x $(0\leqq x\leqq 1)$, K_n takes any subchain of E_{n+1} properly covering q_x onto a subchain of D_n properly covering p_x.

B′. $e_i\subset H_nK_n(e_i)$ for each link e_i of E_{n+1}.

Although we carry along analogues of Condition I of Step 3, the only part of this that will interest us henceforward is the following.

C. If link d_i of D_n intersects p_0+p_1, then $h(d_i\cdot(p_0+p_1))\subset H_n(d_i)$.

C′. If link e_i of E_{n+1} intersects q_0+q_1, then $h^{-1}(e_i\cdot(q_0+q_1))\subset K_n(e_i)$.

Definition of homeomorphism h. For each point p of P let $d(p, i)$ denote the sum of the elements of D_i containing p and $H_id(p, i)$ denote the sum of the images under H_i of the elements of D_i in $d(p, i)$. We note that $d(p, i)$ is the sum of one or two adjacent links of D_i while $H_id(p, i)$ is the sum of one or two adjacent links of E_i.

Of course, $d(p, i+1)\subset d(p, i)$. We now show that $H_{i+1}d(p, i+1)\subset H_id(p, i)$ by showing that if $e_j^{i+1}=H_{i+1}(d_k^{i+1})$ where d_k^{i+1} is an element of D_{i+1} in $d(p, i+1)$, then $e_j^{i+1}\subset H_id(p, i)$. Since $d_k^{i+1}\subset K_iH_{i+1}(d_k^{i+1})$ by Condition B, the link $K_iH_{i+1}(d_k^{i+1})=K_i(e_j^{i+1})$ of D_i lies in $d(p, i)$. Also, $e_j^{i+1}\subset H_iK_i(e_j^{i+1})$ by Condition B′ and $H_iK_i(e_j^{i+1})=H_i(d_k^{i+1})\subset H_id(p, i)$.

For each point p of P, let $h(p)$ be the intersection of the closures of the decreasing sequence of open sets $H_1(d(p, 1))$, $H_2(d(p, 2))$, $H_3(d(p, 3))$, $\cdots$. The intersection exists because $H_{i+1}d(p, i+1))\subset H_i(d(p, i))$. It is a point because the diameter of the closure of $H_i(d(p, i))$ is less than $2/2^i$.

If q is any point of $d(p, i)$, the diameter of $H_i(d(p, i))+H_i(d(q, i))$ is less than $3/2^i$ so $\rho(h(p), h(q))<3/2^i$. Therefore h is continuous.

If $p\in p_x$, we find from Condition A that $H_i(d(p, i))$ intersects q_x. Hence $h(p)\in q_x$.

We now show that h takes P onto Q. Let q be a point of Q and e_j^i be an element of E_i containing q. There is an element d_k^i of D_i such that $H_i(d_k^i)=e_j^i$. Therefore for some point p of P, $e_j^i\subset H_i(d(p, i))$. Since the diameter of $H_i(d(p, i))$ is less than $2/2^i$, $\rho(q, h(p))<2/2^i$. This shows that $h(P)$ is dense in Q. Since $h(P)$ is closed, it is equal to Q.

The transformation h we have defined agrees with the given homeomorphism h on p_0+p_1 because for each point p of p_0+p_1, $H_i(d(p, i))$ contains $h(p)$ by Condition C.

Finally we show that h is 1-1. Suppose $h(p)=h(q)$. Since the closures of $H_id(p, i)$ and $H_id(q, i)$ intersect and we are dealing in this paper only with chains whose nonadjacent links do not have closures that intersect, then $H_id(p, i)$ intersects $H_id(q, i)$. Since K_{i-1} is a chain map, the set $K_{i-1}H_id(p, i)$ intersects the set $K_{i-1}H_id(q, i)$. But the closures of these two sets contain p and q respectively so $\rho(p, q)<4/2^i$. Therefore $p=q$ if $h(p)=h(q)$.

7. **A circle of pseudo-arcs.** It follows from Theorem 10 that each circle of pseudo-arcs is homogeneous and that any two of them are homeomorphic. In this section we show that the plane contains a circle of pseudo-arcs. But we first describe an analogous upper-semicontinuous collection G of arcs in the plane.

Let W_1 and W_2 be circles in the plane with center at (0, 0) and radii equal to one and two, respectively. We shall define a collection $\{g_x \mid -\pi\leqq x\leqq\pi\}$ of mutually exclusive arcs such that each g_x is a straight line interval which is irreducible from W_1 to W_2 (or which is the sum of two such straight line intervals whose intersection is a common end point), $g_{-\pi}=g_\pi$, and $\sum g_x(-\pi\leqq x\leqq\pi)$ is a circle-like continuum.

Let $g_{-\pi}$ be the sum of the two straight line intervals from $(r=1, \theta=-\pi)$ to $(r=2, \theta=-\pi\pm\pi/12)$. Let $g_\pi=g_{-\pi}$. Let g_0 be the sum of the two straight line intervals from (1, 0) to $(2, \pm\pi/12)$. Let $g_{\pi/2}$ be the sum of the straight line intervals from $(2, \pi/2)$ to $(1, \pi/2\pm\pi/12)$ and let $g_{-\pi/2}$ be the sum of the straight line intervals from $(2, -\pi/2)$ to $(1, -\pi/2\pm\pi/12)$. Because of their shape, these elements of $\{g_x\}$ will be called V's.

Now let U denote the set of all points of the annulus W_1W_2 not belonging to any V defined thus far. There are a finite number of components of U which intersect both W_1 and W_2 and in each of these components two V's will be defined. Let the arc T_i $(i=1, 2)$ be the closure of the intersection of such a component of U with W_i. One end point of T_i ($i=1, 2$) is an end point of a V and the other end point of T_i is the vertex of a V. Now construct two disjoint V's in U such that one of these V's has its end points on T_i ($i=1, 2$), the other has its vertex on T_i and these three points are the quarter points of T_i. The reader should note that on W_1 or W_2 no vertex of a V is adjacent to a vertex of a V and no end point of a V is adjacent to an end point of *another* V.

After the above construction of a pair of V's has been carried out for every component of U which intersects both W_1 and W_2, a new U may be defined and the process continued countably many ($\aleph_0$) times. Let G denote the collection of all the V's together with each straight line interval which is the limit of a convergent sequence of V's but which is not a subset of a V. By appropriately choosing the subscripts x, $\{g_x \mid -\pi \leqq x \leqq \pi\}$ is the collection G. It is easy to see that G is an upper-semicontinuous collection of arcs, G is a circle (if its elements are thought of as points) and G^* (the sum of the elements of G) is a circle-like continuum. Something quite similar to the fact that G^* is circle-like has been previously observed by Roberts [25].

The reader will note that G is *not* a *continuous* collection. This is due fundamentally to the fact that in the plane an arc has two sides. Furthermore G^* is not homogeneous for still another reason, namely, some points of G are local separating points (the vertices of the V's) while others are not. But if pseudo-arcs are substituted for the V's, it should be possible to eliminate these two properties from G and G^* respectively while in general keeping G and G^* unchanged in other respects.

As before let W_1 and W_2 be concentric circles in the plane with center at (0, 0) and radii one and two, respectively. Since there are countably many V's of the preceding construction, let them be $V_1, V_2, V_3, \cdots$ in inverse order to the polar angle between a_i and c_i (the end points of V_i). Let b_i be the vertex of V_i and for convenience let b_i belong to W_1 when i is odd but belong to W_2 when i is even.

Now there exists a sequence $D_1, D_2, D_3, \cdots$ of circular chains of simple domains in the plane such that

(1) for each positive integer i, the closure of each element of D_{i+1} is a subset of some element of D_i;

(2) for each i, each element of D_i intersects the annulus W_1W_2 and not both of two intersecting links of D_i intersect W_1+W_2,

(3) if (for each i) δ_i is the maximum diameter of a link of D_i, then $\delta_i \rightarrow 0$,

(4) the subscripts of the elements of D_i which intersect W_1 preserve the counterclockwise order on W_1 and the subscripts of these intersecting W_2 preserve the counterclockwise order on W_2,

(5) if a_i, b_i, and c_i are the end points and vertex of V_i, there is a natural number $m(i)$ such that the (shortest) subchain of $D_{m(i)}$ irreducible from a_i to c_i contains b_i, the subchain of $D_{m(i)+1}$ irreducible from a_i to b_i contains c_i,

the subchain of $D_{m(i)+2}$ irreducible from b_i to c_i contains a_i, the subchain of $D_{m(i)+3}$ irreducible from a_i to c_i contains b_i, etc.,

(6) $(W_1+W_2)\cdot \prod D_i^* =$ closure of $\sum(a_i+b_i+c_i)$ as in the preceding example,

(7) for each i, D_i is the sum of finitely many subchains $T_{i1}, T_{i2}, \cdots, T_{in(i)}$ such that (a) $T_{i1}^*, T_{i2}^*, \cdots, T_{in(i)}^*$ is a circular chain, and (b) for each j, $[1 \leqq j \leqq n(i)]$, T_{ij} is either irreducible from W_1 to W_2 or (for some k) irreducible about $a_k+b_k+c_k$, and

(8) if $h<i$, each element of $\{T_{ij}^*\}_{j=1}^{n(i)}$ is a subset of two intersecting elements of $\{T_{hj}^*\}_{j=1}^{n(h)}$ and

(9) if $h<i$ and T_{ij} is a refinement of $T_{hk}+T_{hl}$, T_{ij} is crooked [3] with respect to $T_{hk}+T_{hl}$ where $l=(k+1) \bmod n(h)$.

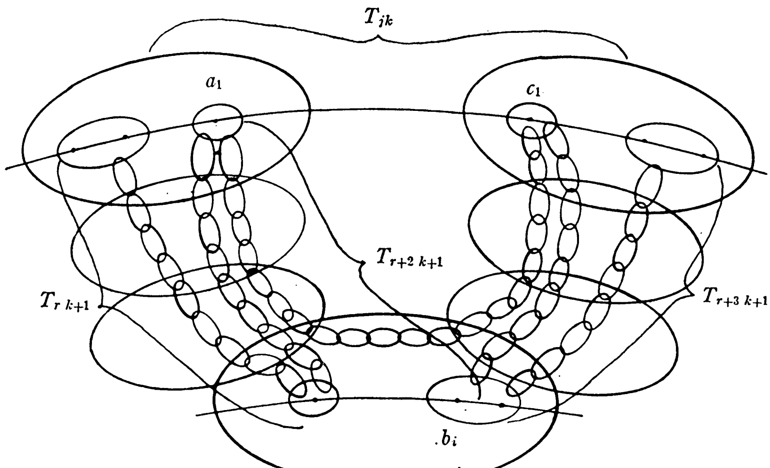


Fig. 3

Now let $M=\prod D_i^*$ and let G denote the set of all subcontinua of M which are irreducible from W_1 to W_2. The reader will observe that if (9) were omitted the construction of the chains D_i could be carried out in such a way that each element of G would be an arc if its intersection with W_1+W_2 were two points and indecomposable if this intersection were three points. Also, in this case M would be a continuous circle-like circle of continua. With the addition of (9) one may see with the help of [3] that each element of G is a pseudo-arc. It follows from Theorem 10 that M is homogeneous.

Bibliography

1. R. D. Anderson, *On monotone interior mappings in the plane*, Trans. Amer. Math. Soc. vol. 73 (1952) pp. 211–222.

2. ———, *Report on results on continuous collections of pseudo-arcs in the plane and homogeneity of the universal curve*, Summary of Lectures and Seminars, Summer Institute on Set Theoretic Topology, Madison, 1955, pp. 76–77.

3. R. H. Bing, *A homogeneous indecomposable plane continuum*, Duke Math. J. vol. 15 (1948) pp. 729–742.

4. ———, *Concerning hereditarily indecomposable continua*, Pacific J. Math. vol. 1 (1951) pp. 43–51.

5. ———, *Snake-like continua*, Duke Math. J. vol. 18 (1951) pp. 653–663.

6. ———, *Another homogeneous plane continuum*, Bull. Amer. Math. Soc. Abstract 60-6-766.

7. ———, *The pseudo-arc*, Summary of Lectures and Seminars, Summer Institute on Set Theoretic Topology, Madison, 1955, pp. 70–73.

8. C. E. Burgess, *Some theorems on n-homogeneous continua*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 136–143.

9. ———, *Certain types of homogeneous continua*, Proc. Amer. Math. Soc. vol. 6 (1955) pp. 348–350.

10. ———, *Homogeneous continua*, Summary of Lectures and Seminars, Summer Institute on Set Theoretic Topology, Madison, 1955, pp. 73–76.

11. Gustav Choquet, *Prolongements d'homéomorphies. Ensembles topologiquement nommables, Caractérization topologique individuelle des ensembles fermés totalement discontinus*, C.R. Acad. Sci. Paris, vol. 219 (1944) pp. 542–544.

12. H. J. Cohen, *Some results concerning homogeneous plane continua*, Duke Math. J. vol. 18 (1951) pp. 467–474.

13. F. B. Jones, *A note on homogeneous plane continua*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 113–114.

14. ———, *Certain homogeneous unicoherent indecomposable continua*, Proc. Amer. Math. Soc. vol. 2 (1951) pp. 855–859.

15. ———, *On a certain type of homogeneous plane continuum*, Bull. Amer. Math. Soc. Abstract 60-6-770.

16. ———, *On homogeneity*, Summary of Lectures and Seminars, Summer Institute on Set Theoretic Topology, Madison, 1955, pp. 66–68.

17. ———, *On a certain type of homogeneous plane continuum*, Proc. Amer. Math. Soc. vol. 6 (1955) pp. 735–740.

18. I. Kapuano, *Sur une proposition de M. Bing*, C. R. Acad. Sci. Paris vol. 236 (1953) pp. 2468–2469.

19. ———, *Sur les continus linéaires*, C. R. Acad. Sci. Paris, vol. 237 (1953) pp. 683–685.

20. B. Knaster, *Un continu dont tout sous-continu est indécomposable*, Fund. Math. vol. 3 (1922) pp. 247–286.

21. B. Knaster and C. Kuratowski, *Problème* 2, Fund. Math. vol. 1 (1920) p. 223.

22. S. Mazurkiewicz, *Sur les continus homogènes*, Fund. Math. vol. 5 (1924) pp. 137–146.

23. E. E. Moise, *An indecomposable plane continuum which is homeomorphic to each of its nondegenerate subcontinua*, Trans. Amer. Math. Soc. vol. 63 (1948) pp. 581–594.

24. ———, *A note on the pseudo-arc*, Trans. Amer. Math. Soc. vol. 64 (1949) pp. 57–58.

25. J. H. Roberts, *Collections filling a plane*, Duke Math. J. vol. 2 (1936) pp. 10–19.

26. Z. Waraszkiewicz, *Sur les courbes planes topologiquement homogenes*, C. R. Acad. Sci. Paris, vol. 204 (1937) pp. 1388–1390.

THE UNIVERSITY OF WISCONSIN,
MADISON, WIS.
THE UNIVERSITY OF NORTH CAROLINA,
CHAPEL HILL, N. C.